Preliminary Exam, Numerical Analysis, August 2012

Instructions: This exam is closed books, no notes and no electronic devices are allowed. You have three hours and you need to work on any three out of questions 1-4, and any three out of questions 5-8. All questions have equal weight and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

Problem 1(**Rank-One Perturbation of the Identity**).
If u and v are n-vectors, the matrix $B = I + uv^\star$ is known as a *rank-one perturbation of the identity*. Show that if B is nonsingular, then its inverse has the form $B^{-1} = I + \beta uv^\star$ for some scalar β, and give an expression for β.
For what u and v is B singular? If it is singular, what is $null(B)$?

Problem 2(**Vector and Matrix Norms**).
a)Let $N(x) := ||\cdot||$ be a vector norm on C^n (or R^n). Show that $N(x)$ is a continuous function of the components $x_1, x_2, ..., x_n$ of x.
In other words, show that

$$x_i \doteq y_i, \quad i = 1, ..., n$$

implies

$$N(x) \doteq N(y)$$

Remark: Notation $\doteq$ means "close to"
b) Give the definition of an induced matrix norm.
c) Explain why $||I|| = 1$ for every induced matrix norm

Problem 3(**Properties via SVD**).
Prove that any matrix in $\mathrm{C}^{m\times n}$ is the limit of a sequence of matrices of full rank. Use the 2-norm for your proof.

Problem 4(**Interpolation**).
Let $x_0, ..., x_n$ be distinct real points, and consider the following interpolation problem. Choose a function

$$F_n(x) = \sum_{j=0}^{n} c_j e^{jx}$$

such that

$$F_n(x_i) = y_i \quad i = 0, 1, ..., n$$

with the $\{y_i\}$ given data. Show there is a unique choice of $c_0, ..., c_n$.

Problem 5(**Unstable Multistep Method**).
Consider the numerical method

$$y_{n+1} = 3y_n - 2y_{n-1} + \frac{h}{2}(f(x_n, y_n) - 3f(x_{n-1}, y_{n-1})), \quad n \geq 1$$

Illustrate with an example of a simple initial value problem that the above scheme is unstable.

Problem 6(**Linear Multistep Methods**).
a) Define linear multistep method (give formula). Give definition of the region of absolute stability.
b) Show that the region of absolute stability for the trapezoidal method is the set of all complex $h\lambda$ with Real(λ)<0.

Problem 7(**Heat Equation and Stability of the Scheme**).
Consider the implicit in time, Backward Euler method for the solution of the heat equation:

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} - a^2 \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2} = f_m^{n+1},$$

$$u_m^0 = g_m, \quad m = 0, \pm 1, \pm 2, ..., \quad n = 0, 1, ..., [T/\Delta t] - 1.$$

and investigate the stability of the scheme using the von Neumann analysis.

Problem 8(**Variational Formulation**).
Consider the two-point boundary value problem

$$-u''(x) = f(x), \quad x \in (0, 1)$$

$$u(0) = u(1) = 0$$

Consider the linear space

$$V = \{v : v \text{ is a continuous function on } [0, 1], v'$$

$$\text{is piecewise continuous and bounded on } [0, 1], \text{ and } v(0) = v(1) = 0\}$$

a) State the variational problem (V) and the minimization problem (M) for the above boundary value problem.
b) Show that the problem (V) and (M) are equivalent.